

# WAITING TO PERSUADE\*

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I analyze a sequential bargaining model in which players are optimistic about their bargaining power (measured as the probability of making offers), but learn as they play the game. I show that there exists a uniquely predetermined settlement date, such that in equilibrium the players always reach an agreement at that date, but never reach one before it. Given any discount rate, if the learning is sufficiently slow, the players agree immediately. I show that, for any speed of learning, the agreement is delayed arbitrarily long, provided that the players are sufficiently patient. Therefore, although excessive optimism alone cannot cause delay, it can cause long delays if the players are expected to learn.

## I. INTRODUCTION

Bargaining delays are common, and frequently cause substantial losses to the bargaining parties. Often, agreements in labor negotiations are reached only after strikes or work slowdowns, and sometimes international conflicts last generations, costing lives and causing lifelong misery. (This might happen while the parties are officially negotiating a peace agreement, as in the case of Israeli-Palestinian conflict.) The usual game-theoretical explanation for these delays is based on asymmetric information: delay is a credible means for a player to communicate his private information that he has a strong position in bargaining, or a screening device to understand whether the other party is in a strong or weak position in bargaining (see Admati and Perry [1987] and Kennan and Wilson [1993]).

There is, however, a sense among researchers that agreement may be delayed even when the parties do not seem to have any asymmetric information about the payoffs. As an alternative cause of bargaining delays, many authors have proposed exces-

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sive optimism due to the lack of a common prior.<sup>1</sup> Based on surveys and experimental and field data, they have concluded that optimism is very common, and have attributed the bargaining delays to excessive optimism. Most of these authors do not have any formal model, but their arguments appear to be based on the two-period negotiation model by Landes [1971] and Posner [1972]. Their reasoning seems to be the following. When each party is excessively optimistic about the share he would get tomorrow, there may not exist any settlement today that satisfies all parties' expectations. In another paper [Yildiz 2003] I have shown that this argument relies critically on the artificial assumption that there are only two periods; in a long horizon model there will be an immediate agreement whenever optimism is sufficiently persistent. The reason is that, if optimism is persistent, then the scope of trade is necessarily small, and thus the players cannot be very optimistic about their share in any agreement in the near future.

This paper provides a new rationale for delay when the parties are optimistic due to the lack of a common prior. Now there is no private information to convey; a player  $i$  simply believes that he has a strong position in bargaining, a belief the other player  $j$  does not share. Being a Bayesian,  $i$  must also believe that the events are likely to proceed in such a way that  $i$  will eventually be proved to have a strong bargaining position (as  $i$ 's beliefs must satisfy the Bayes' rule). In that case,  $j$  will plausibly be convinced that  $i$  is right and thereby be persuaded to agree to  $i$ 's terms. If  $j$ 's initial beliefs are not too firm, this will happen so soon that  $i$  will find it worth waiting to persuade  $j$ . Of course, at the beginning,  $j$  does not believe that the events will proceed in that way; she may even think that  $i$  will be persuaded to agree to *her* terms in the near future. This leads to costly delays that are inefficient even under these optimistic beliefs.

As a formal model, I use the basic model of Yildiz [2003] but focus on the case that the players' initial beliefs are not too firm, allowing them to update their beliefs without restriction. (Yildiz [2003] focuses on the case that the players do not change their beliefs much as they play the game.) In a widely used model of

1. See Hicks [1932], Landes [1971], Posner [1972], Gould [1973], Priest and Klein [1984], Neale and Bazerman [1985], Babcock et al. [1995], and Babcock and Loewenstein [1997]. See also Farber and Bazerman [1989], who show that excessive optimism cannot explain the delay patterns in certain labor negotiations with conventional and final offer arbitration. Some other terms, such as overconfidence and self-serving biases, are also used for what can be called optimism in the present context.

Bayesian learning, I show that there exists a (unique) predetermined date  $t^*$  such that in equilibrium the players will never agree before  $t^*$  and reach an agreement at  $t^*$ . Moreover, they would also have agreed at any date after  $t^*$ , if they had not agreed before. Notice that the settlement date  $t^*$  is common knowledge at the beginning and does not depend on what happens until then. This is surprising, because delay in usual bargaining models—whether caused by signaling, screening, or mixed strategies—is only a possibility, and there is immediate agreement with positive probability. Finding tight bounds for the settlement time  $t^*$ , I further show that delay can be arbitrarily long as long as the players are sufficiently patient. This is true for any initial level of optimism and any firmness of beliefs. Therefore, although optimism alone cannot cause any delay, it can cause delays when it is combined with learning.

The intuition is as above. As is typical in Bayesian learning models, each player  $i$  updates his beliefs relatively quickly at the beginning of the process. When his bargaining partner  $j$  is patient enough, this entices  $j$  to wait so that  $i$  will observe the truth and, hopefully, agree to  $j$ 's terms. After a while, having gained experience through observing some of the data, the players' learning slows down, and each player feels that it is no longer worth waiting for his opponent to change his mind. This is when they reach an agreement. Of course, in the meantime, as the players observe the same data, their beliefs become more similar, and eventually optimism becomes negligible. Nevertheless, the players reach an agreement when the learning slows down—not when the optimism become negligible—as the following two facts establish. First, given any initial beliefs for which there is delay, as we make the players' initial beliefs firmer, the delay becomes shorter and eventually disappears, while learning slows down and the level of optimism *uniformly increases*. Second, there is an upper bound for  $t^*$ —based on the speed of learning—that implies that the agreement is reached long before optimism becomes negligible.

It must be emphasized that there are other models wherein delays are produced from quite different mechanisms. For example, when there are *multiple equilibria*, there can be equilibria with delays, as the equilibrium in continuation game may depend on players' previous actions. Hence, there are usually equilibria with delays in models with simultaneous offers [Perry and Reny 1993; Sakovics 1993] and in models with more than two parties [Baron and Ferejohn 1989]. Incidentally, with more players, even

persistent optimism can cause delays by making the backward-induction process unstable—although the delay is necessarily much shorter than the equilibrium delay if the optimism were not persistent—as Ali [2003] shows. There may also be “*efficient delays*” when the size of the pie is stochastic [Merlo and Wilson 1995; Cripps 1998]. In such an environment the parties may delay the agreement in order to realize a larger pie in the future; inefficiency arises only when they forgo a larger pie in the future by reaching an agreement too early. In addition, if players are allowed to wait for new information before accepting or rejecting an offer, they may exercise their *option value of waiting*, yielding long delays with positive probability [Avery and Zemsky 1994]. Finally, Smith and Stacchetti [2002] also present a continuous-time bargaining model with an endogenous war of attrition; there are two pure-strategy equilibria that yield immediate agreement and a continuum of equilibria in *mixed strategies* that yield delays with positive probabilities.

In the next section I lay out the model and develop the main concepts. The main results are presented in Section III. Section IV extends these results to a continuous-time model. Section V is devoted to a discussion of modeling issues. Section VI concludes. Most of the proofs are presented in a technical appendix, where I develop the notions and the tools that are necessary for these proofs.

## II. MODEL

Take the set of all nonnegative integers  $T = \{0, 1, 2, \dots\}$  as the time space. Take also  $N = \{1, 2\}$  to be the set of players, and  $U = \{u \in [0, 1]^2 | u^1 + u^2 \leq 1\}$  to be the set of all feasible expected utility pairs. Designate dates  $t, s \in T$  and players  $i \neq j \in N$  as generic members.

I will analyze the following perfect-information game. At each  $t \in T$ , Nature recognizes a player  $i \in N$ ;  $i$  offers a utility pair  $u = (u^1, u^2) \in U$ ; if the other player accepts the offer, then the game ends, yielding a payoff vector  $\delta^t u = (\delta^t u^1, \delta^t u^2)$  for some  $\delta \in (0, 1)$ ; otherwise, the game proceeds to date  $t + 1$ . If the players never agree, each gets 0. I assume that the players’ beliefs have beta distributions, a tractable distribution that is widely used in statistical learning models. Fixing any positive integers  $\bar{m}_1, \bar{m}_2$ , and  $n$  with  $1 \leq \bar{m}_2 < \bar{m}_1 \leq n - 2$ , I assume that, for any given dates  $t$  and  $s$  with  $s \geq t$ , at the beginning of date  $t$ , if a

player  $i$  observes that player 1 has made  $m$  offers (and player 2 has made  $t - m$  offers), then he assigns probability

$$(1) \quad (\bar{m}_i + m)/(t + n)$$

to the event that player 1 will make an offer at date  $s$ .<sup>2</sup> This belief structure arises when each player believes that recognition at different dates is identically and independently distributed with some unknown parameter  $\mu$  measuring the probability of player 1 making an offer at any date  $t$ , and  $\mu$  is distributed with a beta distribution with parameters  $\bar{m}_i$  and  $n$  (see a textbook, such as Ross [1998, Chapter 5], for a derivation). I assume that everything described in this paragraph is common knowledge.

As in Yildiz [2003], this model differs from the Rubinstein [1982]-Stahl [1972] framework by allowing different probability distributions for different players. This difference in beliefs about the recognition process can be taken as the difference in beliefs about each player's bargaining power. This is because in sequential bargaining models, including the present one, a player's bargaining power is ultimately determined by the recognition process,<sup>3</sup> rendering the latter a good metaphor for the former, as the following two results suggest. First, Lemma 5 in the Appendix establishes that a player's equilibrium payoff is the present value of all rents he expects to extract when he makes offers in the future. Second, under the assumptions of this paper, a player  $i$  becomes better off in equilibrium whenever each player comes to believe that  $i$  has a higher probability of recognition in the future.

In mediated bargaining environments, the present model is literal if a party needs to be recognized by the mediator before making an offer. It may be a good approximation even if mediation is not of this form. For example, in the Israeli-Palestinian conflict, the negotiations are mediated by the United States. As with the U. S. bridging proposal in December 2000, the mediator may make an offer following deliberations with the parties. This offer can be very close to the offer one of the parties would make, in effect recognizing that party. If the offer is rejected by one of the parties, the conflict continues. After a costly delay, the parties may come back to the table, leading to another offer by the mediator, which can favor one or the other party. Naturally, the parties may have different opinions about how close the mediator

2. This functional form is assumed mainly for tractability; its crucial properties will be discussed later.

3. In contrast, in the mixed strategy equilibria analyzed by Smith and Stacchetti [2001], the players are indifferent toward making an offer.

is to each party, and their beliefs may remain apart even after sharing all information available to them. After observing an offer (and other relevant information that becomes available as time passes), the parties learn about the mediator, and their views about the mediator's future positions probably become closer.<sup>4</sup>

More broadly, the analysis will be similar when there are powerful third parties that may support one of the parties and may shift their support back and forth in a potentially long conflict. These third parties could be the United Nations, the United States, or the European Union in international conflicts, judges in complex cases with multiple subcases, or the general market conditions in labor or trade negotiations. For example, when trading assets for which there is no competitive market, the traders may learn about the value of the future dividends from the current ones as these dividends are likely to be correlated. Likewise, in bargaining between an entrepreneur and a venture capitalist, the movements in the stock and capital markets would have a large impact on each party's bargaining power. In the same vein, when buying a used car from a friend, one may let him first search for other buyers to reach "more realistic" expectations.<sup>5</sup> Even when there is a single event that will determine the parties' eventual bargaining power, the forces in this paper will be in play, as long as the uncertainty about this event is resolved gradually.

On another note, no "true" probability distribution is defined in this paper. Although such a probability distribution could reflect the researcher's beliefs and could be used to assess the likelihood of future events (including the players' future beliefs and play), such an outside probability distribution would be irrelevant for the players' decision problem. This is because each player's beliefs represent his preferences over acts as in Savage [1954], and a player knows his preferences (and hence his beliefs)

4. The beliefs may not become identical after seeing one offer because the offer made at a particular time may depend not only on the inherent position of the mediator but also on the factors that change over time. For example, in December 2000 the parties could not know what the future administration in the United States would be or that there would be terrorist attacks on September 11, 2001.

5. All these examples illustrate alternative models of optimism about bargaining power, where the optimism may be about the inherent value of the objects, which determines the payoffs from both agreement and disagreement, or about the players' outside options. It is not difficult to find examples for yet another model of optimism through the discount rates or the cost of delay for each party. The insights in this model can be easily extended to all these models.

already. A player does take the other player's beliefs into account because these beliefs affect the other player's decision.<sup>6</sup>

**Measuring optimism.** Toward measuring optimism, write  $\Delta = \bar{m}_1 - \bar{m}_2$ . While  $n$  measures the firmness of the players' prior beliefs,  $\Delta/n$  will be shown to measure the initial level of optimism. Notice that the period  $t$  beliefs about the recognition at future period  $s$  depend only on  $t$ —not  $s$ . Hence, optimism will be measured at the time the beliefs are held without distinguishing which future recognition these beliefs are about. Write  $(m, t)$  for the history (at the beginning of date  $t$ ) in which player 1 has made  $m$  offers and player 2 has made  $t - m$  offers. Write  $p_t^i(m)$  for the probability player  $i$  assigns at  $(m, t)$  to the event that he will be recognized at any fixed data  $s \geq t$ . Now, each player  $i$  thinks at  $(m, t)$  that the probability that the other player  $j$  will be recognized at date  $s$  is  $1 - p_t^j(m)$ , while player  $j$  thinks that  $j$  will be recognized with probability  $p_t^j(m)$ , which is higher than  $1 - p_t^i(m)$ , as we will see in a moment. As explained above, this means that player  $i$  thinks that  $j$  is optimistic. Since each player thinks that the other player is optimistic, I will say that the players are *optimistic at  $(m, t)$* . Write

$$y_t(m) = p_t^1(m) + p_t^2(m) - 1$$

for the level of (aggregate) optimism at  $(m, t)$ .<sup>7</sup> By (1),

$$(2) \quad y_t(m) = \frac{\bar{m}_1 - \bar{m}_2}{t + n} \equiv \frac{\Delta}{t + n} > 0.$$

Since  $y_t(m) > 0$ , the players are indeed optimistic at each  $(m, t)$ . Moreover,  $y_t$  is *deterministic*, i.e.,  $y_t$  does not depend on  $m$ ; so  $m$  will be suppressed. This determinism is due to the assumption that the players' beliefs are equally firm; i.e.,  $n$  is same for both players. This will simplify the analysis dramatically.

**Negligible levels of optimism.** As time passes, the players' beliefs merge, and optimism vanishes. Hence, after a while

6. Notice that long-run frequencies are well-defined in this paper and can serve as the "true" probabilities, which the players gradually learn. But once again, the players' beliefs about these frequencies are all that matter—not the researcher's assessments. (See Kreps [1988, Chapter 11] for more discussion.)

7. Given any "true" underlying probability distribution,  $y_t$  measures the aggregate optimism. If the "true" probabilities of recognition for players 1 and 2 at a future date given  $(m, t)$  are  $\hat{p}_{t,1}(m)$  and  $\hat{p}_{t,2}(m)$ , respectively, then the levels of optimism for 1 and 2 are  $y_{t,1}(m) = p_t^1(m) - \hat{p}_{t,1}(m)$  and  $y_{t,2}(m) = p_t^2(m) - \hat{p}_{t,2}(m)$ , yielding  $y_t(m) = y_{t,1}(m) + y_{t,2}(m)$  as the level of aggregate optimism. In relative sense, each player thinks that his opponent's level of optimism is  $y_t(m)$  and he himself is "objective."

the level of optimism will be so low that the players must reach an agreement in equilibrium. A main objective of this paper is to illustrate that delay is mostly caused by learning considerations rather than the high level of optimism. Hence, there is a need for a definition of a negligible level of optimism, a level of optimism such that there can *never* be delay in equilibrium if the player's optimism is at that level or lower, considering all belief structures in the general model of Yildiz [2003]. Now, if the game ends at date 1 (or each player thinks that he will not make any offer after date 1), then the recognized player would take the whole dollar at date 1. In that case, there will be delay whenever  $\delta(p_1^1 + p_1^2) = \delta(1 + y_1) > 1$ , or equivalently whenever  $y_1 > (1 - \delta)/\delta$ , where  $p_1^i$  is the probability  $i$  assigns that he will be recognized at 1 and  $y_1 = p_1^1 + p_1^2 - 1$  is the level of optimism about date 1. One can also show that, for any belief structure, if the level of optimism always remains below  $(1 - \delta)/\delta$ , then there will be immediate agreement. Therefore, I will say that *optimism is negligible* at  $t$  if and only if  $y_t \leq (1 - \delta)/\delta$ . Optimism becomes negligible in this sense at

$$t_0 \equiv \Delta\delta/(1 - \delta) - n.$$

### III. AGREEMENT AND DELAY

For technically oriented readers, in the Appendix I explain the model in greater detail, compute the equilibrium, and develop the tools that are necessary for some of the proofs, which are also relegated to the Appendix. In this section I show that there exists a predetermined  $t^*$  such that the players will never agree before  $t^*$ , and agree at  $t^*$  (and thereafter if they had not yet agreed). Finding very tight bounds for the settlement date  $t^*$ , I show that (i) the agreement can be delayed arbitrarily long, provided that the players are sufficiently patient, and (ii) the agreement is reached when learning slows down, and long before optimism becomes negligible (i.e.,  $t^* < \sqrt{t_0}$ ).

Toward this end, I first present two agreement results in the spirit of Yildiz [2003], who proves similar results under the restrictive assumption that the players do not learn. The first result states that there will be immediate agreement if the optimism is expected to drop slowly.

**THEOREM 1.** For any  $t$  with  $y_t - y_{t+1} \leq (1 - \delta)/\delta$ , there is an agreement regime at  $t - 1$ .

*Proof.* Most proofs are in the Appendix.

That is, in equilibrium there is an immediate agreement as long as it is known that the level of optimism will not drop dramatically in the near future; i.e., as long as the learning will be slow. Theorem 1 implies that the players will agree immediately whenever  $y_1 - y_2 \leq (1 - \delta)/\delta$ . As an immediate corollary, this further implies that if the players' beliefs are sufficiently firm, they will reach an agreement immediately—independent of the initial level of optimism, extending another agreement result in Yildiz [2003]:

**COROLLARY 1.** Let  $\Delta = ny_o$  so that the initial level of optimism remains constant at  $y_o$ . Then, there exists some integer  $\bar{n}$  such that the players reach an agreement immediately whenever  $n \geq \bar{n}$ .

*Proof.* By (2),  $y_1 - y_2 = y_o n / [(n + 1)(n + 2)]$ , which converges to 0 as  $n \rightarrow \infty$ . Therefore, there exists some integer  $\bar{n}$  such that, whenever  $n \geq \bar{n}$ ,  $y_1 - y_2 \leq (1 - \delta)/\delta$ , yielding an immediate agreement by Theorem 1. QED

The main focus of the present paper is on the case that the players' initial beliefs are not firm, and hence they update their beliefs substantially as they observe how players are recognized. In that case, the players may delay the agreement for a while, as the next theorem will imply.

**THEOREM 2.** There exists a  $t^* \in T$  such that, at each  $t \geq t^*$ , the players reach an agreement immediately if they have not reached an agreement yet, and they do not reach an agreement before  $t^*$ .

Theorem 2 establishes that there exists a uniquely predetermined settlement date  $t^*$ . In a moment I will also provide bounds for  $t^*$  and show that  $t^*$  can be arbitrarily large when players are sufficiently patient. In that case, agreement will be delayed. This is because, typically, at the beginning of a learning process players are more open to new information, in the sense that they update their beliefs substantially as they observe which player gets a chance to make an offer. Knowing this, each player waits, believing that the events are very likely to proceed in such a way that his opponent will change his mind. As time passes, they become experienced. In this way, two things occur simultaneously, both facilitating agreement. First, having similar experiences, the discrepancy in their beliefs diminishes. More impor-

tantly, each player  $i$  becomes so closed minded that his opponent  $j$  loses her hope to convince  $i$  and thus becomes more willing to agree to  $i$ 's terms. Therefore, after a while, they reach an agreement. It will be clear that in this process, the latter effect leads the players to an agreement long before the former effect starts playing a role, i.e.,  $y_t - y_{t+1}$  becomes smaller than  $(1 - \delta)/\delta$  long before  $y_t$  does (see the discussion after Lemma 1).

At the beginning of the game it is common knowledge that players will not reach an agreement until  $t^*$ , when they will reach an agreement no matter what happens up to that point.<sup>8</sup> How they will share the pie at  $t^*$  will depend on how many times each player will have been recognized. Since they disagree about how many times each player is likely to be recognized by  $t^*$ , there is no consensus among the players on *how* to make each player better off by agreeing on a decision at the beginning. Therefore, they wait until  $t^*$  even though there *exists* a consensus among them that there is an agreement at the beginning that would leave each player better off.

Two properties of the learning process in (1) are crucial for Theorem 2: (i) the learning slows down monotonically, and (ii) the level of optimism and the speed of learning are deterministic, which is due to the assumption that the initial beliefs of players are equally firm (i.e.,  $n$  is the same for both players). First, if the monotonicity property (i) fails, then we might not have the monotonicity property that players disagree up to a point and agree thereafter. To see this, take parameter values in the present model such that  $t^* > 0$ . Imagine that there have been independently distributed recognitions at times  $\{-\hat{t}, -\hat{t} + 1, \dots, -1\}$  prior to the present negotiation. Hence, players do not learn at all about the future recognitions until  $t = 0$ , when they start learning about the future—fast at 0 and slowing down later. If the players are optimistic about the recognitions at dates  $\{-\hat{t}, -\hat{t} + 1, \dots, -1\}$  and  $\hat{t}$  is sufficiently large, then by Yildiz [2003] they will agree at  $-\hat{t}$ . Since there is disagreement at  $t = 0$  and agreement at  $t^*$ , the monotonicity in Theorem 2 fails. Second, if the speed of learning depends on the past recognitions, then the settlement date  $t^*$  may be stochastic. Nevertheless, when the

8. In contrast, in models with asymmetric information, such as Admati and Perry [1987], delay is only a possibility. In such models, the least advantageous types reach an agreement immediately with positive probability. If all types disagreed with probability 1 in equilibrium, then delay could not be used as a signaling or screening device, as players would adhere to their initial beliefs when agreement is delayed. In the present model, information comes from outside, and hence any delay leads players to change their minds.

players are patient and learn relatively quickly at the beginning, one can still find a deterministic lower bound for delay.

How long will they delay the agreement? To answer this question, the next result provides tight bounds for  $t^*$ .

LEMMA 1. The settlement time  $t^*$  satisfies

$$(3) \quad \max \{0, t_l\} \leq t^* \leq \max \{0, t_u\},$$

where

$$(4) \quad t_u = \frac{\sqrt{1 + 4\Delta\delta/(1 - \delta)} - 1}{2} - n$$

and  $t_l$  is the highest integer  $t$  such that  $s = t + n$  satisfies the cubic inequality,

$$(5) \quad f \equiv (1 - \delta)s(s + 1)(s + 2) - 2(s + 1)\delta\Delta + \delta s\delta\Delta + (\delta\Delta)^2 \leq 0.$$

Lemma 1 provides tight bounds for delay. Here  $t_u$  is computed by equating  $y_t - y_{t+1}$  to  $(1 - \delta)/\delta$ , the cutoff value at which learning becomes negligibly slow. These bounds have two important implications for this paper. First, the upper bound  $t_u$  implies that the players settle long before  $t_0 = \Delta\delta/(1 - \delta) - n$ , when optimism becomes negligible. This is stated in the next result.

THEOREM 3. The settlement time  $t^*$  satisfies

$$t^* \leq \max \{0, t_u\} < \sqrt{t_0}$$

whenever  $t_0 > 0$ .

*Proof.* If  $t_u < 0$ , then the inequality is trivially true. Assume that  $t_u \geq 0$ . By definition,  $t_u + n = \sqrt{t_0 + n + 1/4} - 1/2 < \sqrt{t_0 + n}$ . Hence,  $(t_u + n)^2 < t_0 + n$ , yielding  $t_u^2 < t_0 - (2t_u + n - 1)n < t_0$ . QED

That is, the agreement is reached when the learning slows down, not when optimism becomes negligible. This observation is also supported by the fact that there is immediate agreement when optimism remains always high (cf. Section IV). These suggest that, in reaching an agreement, considerations about learning are more important than optimism itself.

Second, as the players become very patient, the lower bound goes to  $\infty$ , yielding arbitrarily long delays.

**THEOREM 4.** For all  $(t, n, \Delta)$ , there exists  $\bar{\delta} \in (0, 1)$  such that  $t^* \geq t$  whenever  $\delta > \bar{\delta}$ .

*Proof.* Given any  $(t, n, \Delta)$ , since  $n \geq \Delta$ , we have

$$\lim_{\delta \rightarrow 1} f = -\Delta(t + n + 2 - \Delta) < 0,$$

where  $f$  is as defined in (5). Hence, there exists  $\bar{\delta} \in (0, 1)$  such that, whenever  $\delta > \bar{\delta}$ , we have  $f < 0$ , and thus, by Lemma 1,  $t \leq t_l \leq t^*$ . QED

Intuitively, as the players become patient, the efficiency loss due to delay becomes negligible, while each player's individual gain from proving his bargaining power remains substantial, enticing the players to wait arbitrarily long. To see this, consider the limiting case that  $y \equiv 0$ . In that case, the per-period efficiency loss due to delay is  $1 - \delta$ , approaching 0 as  $\delta \rightarrow 1$ . On the other hand, by (8) in the Appendix, the continuation value of a player  $i$  at any  $(m, t)$  is  $p_i^t(m)$ , and hence any increase in  $p_i^t(m)$  is translated to the equilibrium payoff of  $i$ , without vanishing as  $\delta \rightarrow 1$ .

As the players become very patient, although delay becomes arbitrarily long, the efficiency loss due to delay becomes negligible—as Theorem 3 implies.

**COROLLARY 2.** For all  $n$  and  $\Delta$ ,  $\lim_{\delta \rightarrow 1} \delta^{t^*} = 1$ , where  $t^*$  is as defined in Theorem 2.

*Proof.* As  $\delta \rightarrow 1$ ,  $\log \delta^{\sqrt{t_0}} \equiv \sqrt{\Delta/(1 - \delta)} \log \delta \rightarrow 0$ . Hence, by Theorem 3,  $1 \geq \lim_{\delta \rightarrow 1} \delta^{t^*} \geq \lim_{\delta \rightarrow 1} \delta^{\sqrt{t_0}} = 1$ . QED

This corollary is due to the fact that when the players are patient, it costs arbitrarily little to wait until  $t_u \equiv \sqrt{t_0}$ , when learning slows down. In contrast, for patient players, the cost of waiting until  $t_0$  (when optimism becomes negligible) is bounded away from 0. The limit  $\delta \rightarrow 1$  is often interpreted as the continuous-time limit as players are recognized more and more frequently. In that case, Corollary 2 implies that there will be no delay in the continuous-time limit. In such a continuous-time limit, however, optimism also disappears instantaneously, while we are interested in the case that optimism may linger. In a continuous-time model with lingering optimism, the main results of the paper remain intact—as I will show next.

#### IV. DELAY IN A CONTINUOUS-TIME LIMIT

In this section I take a continuum of real times  $\tau$  as the primitive and approximate it with a grid of index-times  $t$ . The

players' time preferences and the level of optimism are given by the real time and do not depend on the grid. Using Lemma 1, I find very simple and intuitive bounds for the real-time limit  $\tau^*$  of the settlement date  $t^*$  as the grid approaches a continuum. I show that the results in the previous section extend to this model: although there is immediate agreement when the optimism is very persistent or instantaneously vanishing, there is delay in between. As the players become sufficiently patient, the real-time delay becomes arbitrarily long.

Taking a continuum of real times  $\tau$ , let the level of optimism at  $\tau$  be

$$(6) \quad y(\tau) = \frac{y_o}{1 + \tau/\pi},$$

where  $y_o$  is the initial level of optimism and  $\pi > 0$  is a parameter measuring the persistence of optimism. Given any  $\tau > 0$ ,  $y(\tau)$  decreases to 0 as  $\pi$  approaches 0, and increases to  $y_o$  as  $\pi \rightarrow \infty$ . Each player's utility from getting  $x$  at  $\tau$  is  $e^{-r\tau}x$ , where  $r > 0$  is the real-time impatience. Now consider a grid of index times  $t$  where each index  $t$  corresponds to a real time  $\tau(t, k) = t/k$ , and  $k > 0$  measures the fineness of the grid. The discount rate is  $\delta(k) = e^{-r/k}$ . Take also  $n = \pi k$  and  $\Delta = y_o n$  so that the level of optimism at a given real time  $\tau(t, k)$  is  $\Delta/(n + t) = y_o/(1 + \tau(t, k)/\pi) = y(\tau(t, k))$  as in (6). Given any  $k$ , let  $t^*(k)$  be the settlement time, defined in Theorem 2 for the parameters  $\delta = e^{-r/k}$ ,  $n = \pi k$  and  $\Delta = y_o n$ . Write also  $\tau^*$  for the limit of  $\tau(t^*(k), k)$  as  $k \rightarrow \infty$ . Building on Lemma 1, the next theorem provides bounds for the real-time delay as the discrete-time grid approaches the continuum.

**THEOREM 5.** In the model of this section, the settlement time  $\tau^* \equiv \lim_{k \rightarrow \infty} \tau(t^*(k), k)$  in the continuous-time limit satisfies

$$\tau^* \leq \tau_u \equiv \max \{ \sqrt{\pi y_o / r} - \pi, 0 \}.$$

Moreover, if  $y_o \pi r < 4/27$ , then

$$\tau^* \geq \sqrt{\frac{\pi y_o}{3r}} - \pi.$$

Finally, given any  $y_o$  and any  $\pi$ ,  $\tau^* \rightarrow \infty$  as  $r \rightarrow 0$ .

First, consistent with the agreement results, the upper bound  $\tau_u$  implies that there is immediate agreement (i.e.,  $\tau^* = 0$ ) whenever  $\pi \geq y_o/r$ ; i.e., when optimism is very persistent. When

$0 < \pi < y_o/r$ , there may be delay, although delay must become arbitrarily short as  $\pi$  approaches 0. The lower bound implies that there will be delay whenever  $0 < \pi < y_o/(3r)$ .

Second, both of the upper and the lower bounds for delay are weakly increasing in  $y_o/r$ . This is intuitive because  $y_o$  scales the speed of learning as well as the level of optimism, while  $r$  measures the players' impatience. The settlement time is determined by when the learning slows down in terms of the players' patience. As  $r$  decreases, the players become patient, increasing the length of delay. As  $r$  approaches 0, the discount rate approaches 1, and  $\tau^*$  goes to infinity for any  $\pi y_o > 0$ , extending Theorem 4 to the present setup. Therefore, in the continuous-time limit, there will be very long real-time delays if the players are patient, optimistic, *and* can learn about their bargaining power in the process of bargaining.

## V. MODELING ASSUMPTIONS AND METHODOLOGY

This paper makes several modeling assumptions in order to illustrate a rationale for bargaining delays based on differences in players' beliefs. In particular, it relies on equilibrium and assumes that there is no asymmetric information. In this section I will discuss these and some other assumptions in more detail, and explain how the results would change if some of these assumptions were dropped.

Let me start with the equilibrium and the rationality assumptions behind it. First, the (subgame-perfect) equilibrium in this paper uses simple Markovian strategies that only depend on how many times each player has been recognized. Hence playing equilibrium does not require sophisticated contingent planning. Moreover, all nonequilibrium strategies are ruled out by iterated elimination of conditionally dominated strategies, and hence equilibrium is *implied by* common strong belief in sequential rationality [Battigalli and Siniscalchi 2002].<sup>9</sup> That is, without this (strong) rationality assumption, some other strategies will also be possible outcomes, but each such strategy will be ruled out as we make more and more knowledge assumptions about sequential rationality. In particular, if we assume that the players

9. This shows that the present paper is immune to the critique of Dekel, Fudenberg, and Levine [forthcoming], who point out the difficulties in interpreting equilibrium as a limit of a learning process when there are large differences in players' beliefs about Nature's moves. Also, for iterated conditional dominance, see Fudenberg and Tirole [1991] and Shimoji and Watson [1998].

will act sequentially rationally in the next  $k$  dates, that players are certain that the other players will act sequentially rationally in the next  $k$  dates, that players are certain that players are certain that . . . up to the order  $k$ , then we necessarily conclude that each player's payoff is in the  $\delta^k$  neighborhood of his equilibrium payoff. (See the proof of Theorem 1 in Yildiz [2003].)

There is no inconsistency in using rationality assumptions without the common-prior assumption, as these two assumptions are not related [Savage 1954; Chapter 3]. Combined, they become quite restrictive [Aumann 1976; Milgrom and Stokey 1982], and, not surprisingly, most criticism toward the common prior assumption comes from the authors who take a rationalistic approach, such as Savage [1954], Kreps [1988, 1991], Morris [1995], and Gul [1998]. The arguments in favor of the common-prior assumption are that it simplifies analysis, that interplayer differences can be modeled within this framework [Harsanyi 1968], that it is widely used in economics [Aumann 1987], and that it allows us to focus on informational issues [Aumann 1987]. On the other hand, there are several studies, such as Harrison and Kreps [1979], Morris [1996], Banerjee and Somanathan [2001], and Van den Steen [2001], that suggest that we can obtain valuable insights about differences in players' beliefs in very simple models by dispensing with the common-prior assumption. Such a modeling strategy allows us to focus on differences in beliefs without getting drowned in the informational issues. Finally, as in the present model, players' beliefs about the future frequencies may merge after observing a long sequence of data, making the common-prior assumption a good approximation for such players after they share a very long common past. But this paper shows that the anticipation of such an eventuality will tend to cause bargaining delays at the beginning of their relationship.

How can one justify assuming that there is substantial discrepancy between the players' initial beliefs when the players can learn and have lived in the same world prior to the interaction? One possible answer lies in the size of the signal space. Although there are sufficient conditions for merging [Blackwell and Dubins 1962], these conditions become extremely onerous when there are infinitely many possible values the signals can take. In the latter case, Freedman [1965] shows that the set of parameters under which two players' beliefs merge is of Category I (i.e., a countable union of nowhere-dense sets). Now, the parties' general life experiences are about a much broader world compared with their learning experience about their bargaining power in the present

negotiation. Clearly, the range of signal space associated with the broader world is much larger than that with the narrow interaction. Therefore, while Freedman's nonconvergence result definitely applies to the broader life experiences, Blackwell and Dubins' result *may* apply to their learning in a narrow interaction.

I will now discuss the role of the assumption that the players' beliefs are common knowledge. Observe first that, in sequential bargaining, a player's beliefs about the recognition process constitute his bargaining power when they are common knowledge. That is, a player's equilibrium payoff increases as he becomes more optimistic. In particular, if it becomes common knowledge that at each date he has the extreme belief that he will make all the remaining offers independent of the past recognitions, his equilibrium payoff will be the highest. When the other player's beliefs are sufficiently firm, there is already an immediate agreement, and hence the player's actual share is the highest when he holds such extreme beliefs.<sup>10</sup> When the other player's beliefs are not firm, it is natural to expect that he would still gain from his perceived optimism (under an "objective" probability distribution), as his opponent would give in early on when she faces such optimism. It is also natural to expect that an optimistic player's expected payoff increases as his beliefs become firmer, as that deters his opponent from trying to convince him. Now, when a player's beliefs are not common knowledge, they become his private information, as he knows his own beliefs. Hence, dropping the common knowledge assumption leads to a bargaining model with incomplete information about the players' bargaining power. In that case, one would expect that the players try to form a reputation for being optimistic and a "firm believer," while their opponents try to distinguish nonoptimistic or agnostic players from the others, leading to common patterns of signaling and screening. Ali and Yildiz [2003] analyze such a model without learning—as in Yildiz [2003]. Although it is far beyond the means of the present simple model to establish formally, it suggests that in a model with learning, the learning considerations will be

10. Hence, it is a Nash equilibrium that players choose to hold these extremely optimistic beliefs in a model in which the players choose their "observable" beliefs before the bargaining. Assuming that this is indeed the only Nash equilibrium, the usual evolutionary models of preferences would predict extreme optimism in the limit—as opposed to recognizing the underlying frequencies in the model. Although such models are common (and predict optimism in similar environments), they heavily rely on equilibrium or rationality, which lose their evolutionary appeal in these environments [Acemoglu and Yildiz 2001]. Freedman's result also applies here.

there, entangled with screening and signaling motives, all positively contributing to a long delay.

## VI. CONCLUSION

This paper presents a new rationale for bargaining delays based on optimism and learning. It observes that when two optimistic, Bayesian players negotiate, each player  $i$  believes that the events are likely to proceed in such a way that  $i$  will eventually be proved right. If the other player  $j$ 's initial beliefs are not too firm, this will entice  $i$  to wait in the hopes that  $j$  will quickly learn that  $i$  is right and thereby be persuaded to agree to  $i$ 's terms. This yields costly delays that may be arbitrarily long and are inefficient even under these optimistic beliefs. In this reasoning the considerations about learning seem to be more salient than optimism itself. In fact, in equilibrium the players will settle when the players' learning slows down, and long before optimism becomes negligible. Moreover, they will settle immediately whenever the level of optimism is expected to remain high for a long while. In conclusion, although excessive optimism alone cannot cause delays, it can cause long delays when the players are expected to learn in the future.

## APPENDIX: A MORE TECHNICAL EXPOSITION

In this appendix I first explain some details of the model, describe the subgame-perfect equilibria (henceforth, simply equilibria), and establish the relationship between the recognition process and players' bargaining power. I then find a condition that determines whether there is agreement at a given history  $(m, t)$ . Next, I find tight bounds for equilibrium payoffs. These bounds allow me to prove the results that have not yet been proved. I will write  $E^i(\cdot | m, t)$  for the conditional-expectation operator for player  $i$  at history  $(m, t)$  throughout.

We have a usual game tree (the same for both players) with the important exception that for each move of Nature, there are two probabilities—rather than one—representing the beliefs of the two players. At the first node at each date  $s$ , Nature moves, recognizing either player 1 or player 2. The probabilities for the move that recognizes player 1 are  $p_s^1(m')$  and  $1 - p_s^2(m')$ , representing the beliefs of players 1 and 2, respectively, where  $m'$  is the number of times player 1 has been recognized. Since they are functions of  $m'$ ,  $p_s^1$  and  $p_s^2$  are random variables. At any history

$(m, t)$  with  $t < m$ , a player  $i$  assigns probability  $p_t^i(m)$  to the event that he is recognized at date  $s$ . Notice that this probability is independent of  $s$  as discussed earlier. By Bayes' rule,  $p_t^i(m) = E^i(p_s^i|m, t)$ ; i.e.,  $p_t^i(m)$  is the weighted average of his probability assessments  $p_s^i(m')$  at the nodes  $(m', s)$  that follow  $(m, t)$ .

The first result is taken from Yildiz [2003]; it extends the usual uniqueness result in the Rubinstein [1982]-Stahl [1972] framework to the present model.

LEMMA 2. Given any  $(m, t, i)$ , there exists a unique  $V_t^i(m) \in [0, 1]$  such that, in any subgame-perfect equilibrium, the continuation value of  $i$  at  $(m, t)$  is  $V_t^i(m)$ .

That is, there is a unique equilibrium continuation value at each history, yielding a well-defined social surplus

$$S_t = V_t^1 + V_t^2$$

at each date  $t$ . While the continuation values of each player will in general depend on the realized  $m$ , the social surplus  $S_t$  is deterministic.<sup>11</sup>

LEMMA 3. For each  $t$ ,  $S_t$  is deterministic; i.e.,  $S_t(m) = S_t(m')$  for all  $m$  and  $m'$ .

*Proof.* (Sketch—see Yildiz [2001] for a complete but tedious proof.) The infinite-horizon game here can be truncated at some  $\bar{t}$ , by assigning  $(0, 0)$  as the payoff vector at  $\bar{t}$ . Moreover, it can be seen from Lemma 5 below that, if  $S_s$  is deterministic for each  $s > t$ , so is  $S_t$ . By induction,  $S_t$  must be deterministic in the truncated game. But the equilibrium payoff vector in the infinite-horizon game (namely  $V$ ) is the limit of the equilibrium payoff vectors of the truncated games (as  $\bar{t} \rightarrow \infty$ ). Hence, letting  $\bar{t} \rightarrow \infty$ , one obtains the lemma. QED

If the discounted value of next period's surplus is less than 1 (i.e.,  $\delta S_{t+1} \leq 1$ ), then there are mutually perceived gains to be made from immediate agreement—I call this an *agreement regime*. The currently recognized player has all of the bargaining power on this gain, and thus he extracts entire *rent*  $R_t = 1 - \delta S_{t+1}$  in an agreement regime. (The recognized player  $i$  gets  $1 - \delta V_{t+1}^i = R_t + \delta V_{t+1}^i$ , leaving the other player  $j$  only  $\delta V_{t+1}^j$ .) If  $\delta S_{t+1} > 1$ , there cannot be any agreement at  $t$  that satisfies both

11. This very helpful lemma crucially relies on the assumption that players' initial beliefs are equally firm.

players' expectations, and hence they disagree at  $t$ —I call this a *disagreement regime*. There is no rent in that case; i.e.,  $R_t = 0$ . Since  $S_t$  is deterministic, the rent,

$$R_t = \max \{1 - \delta S_{t+1}, 0\},$$

is also deterministic. Hence, the continuation value of a player  $i$  at the beginning of  $t$  can be expressed in terms of the rent at  $t$  and the discounted expected value (using his own beliefs) of his continuation value at the beginning of the next date, as in the next lemma.<sup>12</sup>

LEMMA 4. Given any  $(m, t)$  and  $i$ ,

$$(7) \quad V_t^i(m) = p_t^i(m)R_t + \delta E^i(V_{t+1}^i | m, t).$$

By iterated application of (7), one can then express the current continuation value as an infinite sum over expected future rents, i.e.,  $V_t^i(m) = \sum_{s=t}^{\infty} \delta^{s-t} E^i(p_s^i R_s | m, t)$ . Because of the belief structure (namely  $E^i(p_s^i | m, t)$  is independent of  $s$ ) and determinism of the rents, this infinite sum simplifies greatly, allowing a simple expression for the continuation values:

LEMMA 5. Given any  $(m, t)$  and  $i \in N$ ,

$$(8) \quad V_t^i(m) = p_t^i(m)\Lambda_t$$

$$(9) \quad S_t = (1 + y_t)\Lambda_t,$$

where

$$(10) \quad \Lambda_t = \sum_{s=t}^{\infty} \delta^{s-t} R_s$$

is the present value of all future rents.

*Proof.* By the law of iterated expectations, the solution to the difference equation (7) is

12. This lemma is a straightforward generalization of a basic result in the Rubinstein [1982]-Stahl [1972] framework. To see the proof, consider  $\delta S_{t+1} \leq 1$ , and assume that Player 1 is recognized. Since  $1 - \delta V_{t+1}^2(m+1) \geq \delta V_{t+1}^1(m+1)$ , Player 1 offers  $(1 - \delta V_{t+1}^2(m+1), \delta V_{t+1}^2(m+1))$ , which is barely accepted. Likewise, if Player 2 is recognized, they agree on  $(\delta V_{t+1}^1(m), 1 - \delta V_{t+1}^2(m))$ . The continuation value of Player 1 at  $(m, t)$  is thus

$$\begin{aligned} V_t^1(m) &= p_t^1(m)(1 - \delta V_{t+1}^2(m+1)) + (1 - p_t^1(m))\delta V_{t+1}^1(m) \\ &= p_t^1(m)(1 - \delta S_{t+1}) + \delta E^1(V_{t+1}^1 | m, t) = p_t^1(m)R_t + \delta E^1(V_{t+1}^1 | m, t). \end{aligned}$$

Similarly,  $V_t^2(m) = p_t^2(m)R_t + \delta E^2(V_{t+1}^2 | m, t)$ . When  $\delta S_{t+1} > 1$ , the players cannot agree at  $t$ , and thus  $V_t^i(m) = \delta E^i(V_{t+1}^i | m, t)$ .

$$V_t^i(m) = \sum_{s=t}^{\infty} \delta^{s-t} E^i(p_s^i R_s | m, t) = \sum_{s=t}^{\infty} \delta^{s-t} p_t^i(m) R_s = p_t^i(m) \Lambda_t,$$

where the second equality is due to the fact that  $R_s$  is deterministic and that  $E^i(p_s^i | m, t) = p_t^i(m)$ . Summing up (8) over the players, one obtains (9). (See Yildiz [2000] for details.) QED

Equation (8) states that the continuation value of a player  $i$  at any  $(m, t)$  is simply the probability  $p_t^i(m)$  that he gets recognized multiplied by the present value  $\Lambda_t$  of all the future rents. This yields the simple expression for  $S_t$  in (9). Equation (9) has two immediate corollaries that are crucial for this paper. First, there is an agreement regime at any  $t - 1 \in T$  if and only if

$$(11) \quad \Lambda_t \leq 1/(\delta + \delta y_t) \equiv D_t.$$

Second, in any agreement regime,  $\Lambda_t$  satisfies a simple difference equation:

LEMMA 6. For any  $t$ , if  $\delta S_{t+1} \leq 1$ , then  $\Lambda_t = 1 - \delta y_{t+1} \Lambda_{t+1}$ .

*Proof.* Assume that  $\delta S_{t+1} \leq 1$ . Then,  $R_t = 1 - \delta S_{t+1}$ . Hence (9) yields  $R_t = 1 - \delta(1 + y_{t+1})\Lambda_{t+1}$ . Hence, by (10),  $\Lambda_t = R_t + \delta\Lambda_{t+1} = 1 - \delta y_{t+1} \Lambda_{t+1}$ . QED

It is this difference equation that yields the bounds that constitute the key technical portion of this paper. Since  $|\delta y_{t+1}| < 1$ , the difference equation is stable (backward in time), and thus one can obtain bounds for  $\Lambda_t$  by solving the difference equation  $\Lambda_t = 1 - \delta y_{t+1} \Lambda_{t+1}$  forward. Although the details appear complicated, the intuition is simple: if  $y_{t+1}$  were a constant  $\bar{y}$  for each  $t$  (which is approximately true for large  $t$ ), then Lemma 6 would yield  $\Lambda_t = 1/(1 + \delta\bar{y})$ , as this is the global attractor for the difference equation  $\Lambda_t = 1 - \delta\bar{y}\Lambda_{t+1}$ . I will show that, in fact,

$$\bar{B}_{t-1} < \Lambda_t < \bar{B}_t \quad (\forall t \in PA),$$

where

$$(12) \quad \bar{B}_t \equiv 1/(1 + \delta y_{t+1})$$

and

$$(13) \quad PA \equiv \{t \in T | \Lambda_s \leq D_s \forall s > t\}$$

is the interval of the dates  $t$  such that there is an agreement regime at each  $s \geq t$ . The upper bound  $\bar{B}_{t+1}$  for  $\Lambda_{t+1}$  and Lemma 6 yield a tighter lower bound for  $\Lambda_t$ :

$$(14) \quad \Lambda_t > 1 - \delta y_{t+1} \bar{B}_{t+1} \equiv \underline{B}_t \quad (\forall t \in PA).$$

Notice that  $\underline{B}_t = (1 - \delta(y_{t+1} - y_{t+2})) / (1 + \delta y_{t+2})$ . Notice also that these bounds are very tight, as  $\bar{B}_{t-1} < \underline{B}_t < \bar{B}_t$ , and are valid only on  $PA$  as the recursive equation above is true only at agreement regimes. (See Figure I for illustration.)

In what remains of the Appendix, I will first establish these bounds, and then use these bounds in presenting the remaining proofs. In particular, the upper bound allows me to prove Theorem 1: I simply compare the upper bound  $\bar{B}_t$  with  $D_t$ , the cutoff value for  $\Lambda_t$  that determines whether there is an agreement regime at  $t - 1$ . Since this comparison is equivalent to the comparison of  $y_t - y_{t+1}$  to  $(1 - \delta)/\delta$ , and since that difference is decreasing, the smallest element in  $PA$  must be smaller than  $t_u$ , the unique solution to the equation  $y_t - y_{t+1} = (1 - \delta)/\delta$ . It turns out that this yields Theorem 2: disagreement can only occur when  $y_t - y_{t+1} > (1 - \delta)/\delta$  and  $\delta S_{t+1} > 1$ . But as I show, these together imply that  $S_t > S_{t+1} > 1/\delta$ . So whenever there is a disagreement regime at  $t$ , there is a disagreement regime at

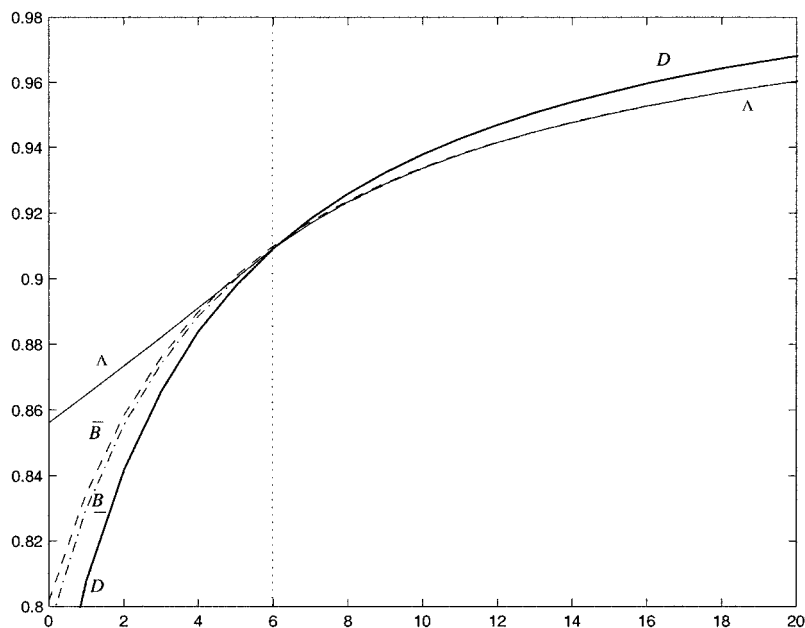


FIGURE I

Functions  $D$ ,  $\underline{B}$ ,  $\bar{B}$ , and  $\Lambda$  ( $\delta = 0.99$ ,  $n = 3$ ,  $\Delta = 1$ ;  $t_l = t^* = 6$ , and  $t_u \cong 6.46$ .)

$t - 1$ . Hence,  $PA$  coincides with agreement regimes, and I can compare the lower bound  $\underline{B}_t$  for  $\Lambda_t$  with  $D_t$  and obtain the lower bound  $t_l$  in Lemma 1. Finally, I prove Theorem 5 with some algebra.

To establish the bounds for  $\Lambda_t$ , I need one more piece of notation; define  $C_t \equiv (1 - \bar{B}_{t-2})/(\delta y_t) = y_{t-1}/[y_t(1 + \delta y_{t-1})] > \bar{B}_t$ , so that

$$(15) \quad \bar{B}_{t-1} = 1 - \delta y_{t+1} C_{t+1}.$$

In a moment I will prove the following.

LEMMA 7. For any  $t \in PA$ ,  $\bar{B}_{t-1} < \Lambda_t < C_t$ .

This immediately yields the desired bounds.

LEMMA 8. For each  $t \in PA$ ,  $\underline{B}_t < \Lambda_t < \bar{B}_t$ .

*Proof.* Take any  $t \in PA$ . Notice that, if  $\Lambda_{t+1} > \bar{B}_t$ , then by Lemma 6,  $\Lambda_t < 1 - \delta y_{t+1} \bar{B}_t = \bar{B}_t$ . Likewise, if  $\Lambda_{t+1} < C_{t+1}$ , then  $\Lambda_t > 1 - \delta y_{t+1} C_{t+1} = \bar{B}_{t-1}$ . But  $t + 1 \in PA$ , and hence by Lemma 7, we have  $\bar{B}_t < \Lambda_{t+1} < C_{t+1}$ , showing that  $\bar{B}_{t-1} < \Lambda_t < \bar{B}_t$ . Finally,  $\Lambda_{t+1} < \bar{B}_{t+1}$  implies (by Lemma 6) that  $\Lambda_t > 1 - \delta y_{t+1} \bar{B}_{t+1} = \underline{B}_t$ . QED

Toward proving Lemma 7, use Lemma 6 to obtain

$$(16) \quad \Lambda_{t+1} = \bar{B}_t - b \Leftrightarrow \Lambda_t = \bar{B}_t + \delta y_{t+1} b,$$

$$(17) \quad \Lambda_{t+1} = C_{t+1} + c \Leftrightarrow \Lambda_t = \bar{B}_{t-1} - \delta y_{t+1} c$$

for each  $t \in PA$  and  $b, c \in \mathbb{R}$ .

*Proof of Lemma 7.* Take any  $t \in PA$ , and define  $\theta_t^s$  by setting  $\theta_t^t = 1$  and  $\theta_t^s = \prod_{k=t+1}^s (-\delta y_k)$  at each  $s > t$ . Using (16) and (17), and mathematical induction on  $l$ , one can check that

$$(18) \quad \Lambda_t = C_t + \theta_t^{t+2l} [\Lambda_{t+2l} - C_{t+2l}] - \sum_{0 \leq k \leq l-1} \theta_t^{t+2k} [C_{t+2k} - \bar{B}_{t+2k}]$$

for each  $t \in PA$ , and  $l \geq 0$ , where I use the convention that a sum over the empty set is 0.

Equation (18) implies that  $\Lambda_t < C_t$  when  $l$  is sufficiently large. To see this, note first that, since  $|\delta y_t| < 1$ , as  $l \rightarrow \infty$ ,  $\theta_t^{t+2l} \rightarrow 0$ . Since  $|\Lambda_{t+2l} - C_{t+2l}| < 1$  at each  $t, l$ , it follows that, as  $l \rightarrow \infty$ ,  $\theta_t^{t+2l} [\Lambda_{t+2l} - C_{t+2l}] \rightarrow 0$ . Second,  $\theta_t^{t+2k} > 0$  for each  $k$ , as it consists of multiplication of evenly many negative numbers. Since  $C_{t+2k} - \bar{B}_{t+2k}$  is always positive, it follows that  $\sum_{0 \leq k \leq l-1} \theta_t^{t+2k} [C_{t+2k} - \bar{B}_{t+2k}]$  is positive, increasing in  $l$ , and,

hence, bounded away from 0. Therefore, there exists a nonnegative integer  $l'$  such that

$$(19) \quad \theta_t^{t+2l}[\Lambda_{t+2l} - C_{t+2l}] - \sum_{0 \leq k \leq l-1} \theta_t^{t+2k}[C_{t+2k} - \bar{B}_{t+2k}] < 0$$

whenever  $l \geq l'$ , whence  $\Lambda_t < C_t$  by (18).

On the other hand, using (18) at  $t + 1$  and (17), one can also obtain

$$(20) \quad \Lambda_t = \bar{B}_{t-1} - \delta y_{t+1} \theta_{t+1}^{t+2l+1}[\Lambda_{t+2l+1} - C_{t+2l+1}] \\ + \delta y_{t+1} \sum_{0 \leq k \leq l-1} \theta_{t+1}^{t+1+2k}[C_{t+1+2k} - \bar{B}_{t+1+2k}].$$

Of course, by (19), there exists some nonnegative integer  $l''$  such that

$$(21) \quad -\delta y_{t+1} \theta_{t+1}^{t+2l+1}[\Lambda_{t+2l+1} - C_{t+2l+1}] \\ + \delta y_{t+1} \sum_{0 \leq k \leq l-1} \theta_{t+1}^{t+2k+1}[C_{t+2k+1} - \bar{B}_{t+2k+1}] > 0$$

whenever  $l \geq l''$ , whence  $\Lambda_t > \bar{B}_{t-1}$  by (20). Therefore, for any  $l \geq \max\{l', l''\}$ , inequalities (19) and (21) simultaneously hold. Hence, by (18) and (20),  $\bar{B}_{t-1} < \Lambda_t < C_t$ . QED

*Proof of Theorem 1.* First observe that, by definition,

$$(22) \quad \bar{B}_t \leq D_t \Leftrightarrow y_t - y_{t+1} \leq (1 - \delta)/\delta,$$

where  $D_t$ , the cutoff value for  $\Lambda_t$  that determines whether there is an agreement, and the upper bound  $\bar{B}_t$  for  $\Lambda_t$  are defined in (11) and (12), respectively. Since  $y_t - y_{t+1}$  is decreasing in  $t$  and approaches 0 as  $t \rightarrow \infty$ , there exists some real number  $t_u$  such that  $\bar{B}_t \leq D_t$  if and only if  $t \geq t_u$ . Now, assume that  $y_t - y_{t+1} \leq (1 - \delta)/\delta$ . Then,  $t \geq t_u$ , and hence  $\Lambda_s < \bar{B}_s \leq D_s$  for each  $s \geq t$ , showing (by (13)) that  $t - 1 \in PA$ ; i.e., there is an agreement at  $t - 1$ . QED

*Proof of Theorem 2.* Take  $t^* \equiv \min PA$ . (Notice that  $PA \neq \emptyset$ .) By definition, there is an agreement regime at each  $t \geq t^*$ , and hence it suffices to show that there is a disagreement regime at each  $t < t^*$ . If  $t^* = 0$ , this is vacuously true, so assume that  $t^* > 0$ . In that case,  $t^* - 1 < t_u$ , and there is a disagreement regime at  $t^* - 1$ . Now I will show that, whenever there is a disagreement regime at any  $t < t_u$ , there will also be a disagreement regime at  $t - 1$ , showing by mathematical induction that

there is a disagreement regime at each  $s \leq t^* - 1$ . To this end, take some  $t < t_u$  with a disagreement regime so that the social surplus at  $t + 1$  is large:  $S_{t+1} > 1/\delta$ . Since there is no rent at  $t$  (i.e.,  $R_t = 0$ ), the present value  $\Lambda_t$  of the future rents satisfies  $\Lambda_t = \delta\Lambda_{t+1}$ . By (9), this yields

$$S_t = (1 + y_t)\delta\Lambda_{t+1} = S_{t+1}\delta(1 + y_t)/(1 + y_{t+1}).$$

Hence,  $S_t \geq S_{t+1}$  whenever  $\delta(1 + y_t) \geq 1 + y_{t+1}$ , i.e., whenever  $y_t - y_{t+1}/\delta \geq (1 - \delta)/\delta$ . But this is true:  $y_{t+1} \geq 0$  and  $t < t_u$ , hence  $y_t - y_{t+1}/\delta \geq y_t - y_{t+1} \geq (1 - \delta)/\delta$ . Therefore,  $S_t \geq S_{t+1} > 1/\delta$ , and hence there is a disagreement regime at  $t - 1$ . QED

*Proof of Lemma 1.* The upper bound  $t_u$  is computed by setting  $\Delta/[(t_u + n)(t_u + n + 1)] = (1 - \delta)/\delta$ . Since  $y_t - y_{t+1} = \Delta/[(t + n)(t + n + 1)]$ , (22) yields  $t^* \leq \max\{0, t_u\}$ . To compute  $t_l$ , use the lower bound  $\underline{B}_t$  for  $\Lambda_t$ . Check that  $\underline{B}_t \geq D_t$  if and only if  $t$  satisfies (5). By Lemma 8,  $s = t + n$  satisfies (5) only if  $t \leq t_u$ , hence there exists a largest integer  $t_l$  that satisfies (5), and  $t_l \leq t_u$ . Clearly,  $\Lambda_{t_l} > \underline{B}_{t_l} > D_{t_l}$ . Hence, by (11) there is disagreement at  $t_l - 1$ . Then,  $t_l - 1 < t^*$ , and therefore  $t_l \leq t^*$ . QED

*Proof of Theorem 5.* Given any  $k$ , let  $t^*(k)$  and  $t_u(k)$  be the settlement time and its upper bound, respectively, defined in Lemma 1 for the parameters  $\delta = e^{-r/k}$ ,  $n = \pi k$ , and  $\Delta = y_o n$ . Write  $\tau^*$  and  $\tau_u$  for the limits of  $\tau(t^*(k), k)$  and  $\tau(t_u(k), k)$ , respectively, as  $k \rightarrow \infty$ . Define  $\omega = \pi y_o/r$ .

Toward proving the first statement, note that  $\delta = e^{-r/k} \cong 1 - r/k$  for large values of  $k$ . Hence, by (4),  $t_u(k) \cong (\sqrt{1 + 4\omega k^2} - 1)/2 - \pi k$  so that  $\tau_u = \lim_{k \rightarrow \infty} t_u(k)/k = \sqrt{\omega} - \pi$  as claimed. To prove the second statement, for any given  $t$ , write  $s \equiv t + n$  and  $\sigma \equiv s/k = \tau(t, k) + \pi$ . When  $k$  is large, we also have  $\delta = e^{-r/k} \cong 1 - r/k$  and  $(k\sigma + 1)/k \cong \sigma \cong (k\sigma + 2)/k$ . Substituting these in (5), one can check that  $f \cong rk^2[\sigma^3 - \omega\sigma + \omega^2r]$ . Then, by Theorem 1,  $\tau^* \geq \sigma - \pi$  whenever

$$\phi \equiv \sigma^3 - \omega\sigma + \omega^2r \leq 0.$$

Note that  $\phi$  has a local minimum at  $\underline{\sigma} = \sqrt{\omega/3}$ . If  $y_o\pi r < 4/27$ , then  $\phi$  is negative at  $\underline{\sigma}$ , showing that  $\tau^* > \underline{\sigma} - \pi$  as desired.<sup>13</sup> The

13. If  $\tau_l$  is the real-time limit of the lower bound in Lemma 1 as  $k \rightarrow \infty$ , then  $\tau_l + \pi$  is the largest solution to the cubic equation  $\phi = 0$ , which is greater than  $\underline{\sigma}$ . When  $y_o\pi r > 4/27$ ,  $\phi$  has a unique root, which is negative.

last statement in the theorem follows from the fact that as  $\pi \rightarrow 0$ ,  $y_o \pi r \rightarrow 0$  and thus  $\tau^* > \sqrt{\omega/3} - \pi \rightarrow \infty$ . QED

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